

Linking Armington and CET Elasticities of Substitution and Transformation to Price Elasticities of Import Demand and Export Supply: A Note for CGE Practitioners

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1. INTRODUCTION

Armington and Constant-Elasticity-of-Transformation (CET) functions are routinely used in Computable General Equilibrium (CGE) Models to model foreign trade. The CET function is applied to producer decisions about whether to export and or sell at home. Decisions by domestic demanders whether to purchase imports or domestic output is covered by the Armington function. The Armington function is a Constant-Elasticity-of-Substitution (CES) function, referred to as an Armington function in honor of the economist Paul Armington, who proposed the use of a CES function in this context (see Armington, 1969). The CET function was first developed by Powell and Gruen (1968).

This note is concerned with the link between

- price elasticities of import demand and constant elasticities of substitution between demand for imports and domestic output; and
- price elasticities of export supply and constant elasticities of transformation between supply to exports and domestic market

This is important since researchers often may wish to compare estimates of price elasticities to the price elasticities implied by elasticities of substitution and transformation.

In the following note, price elasticities related to each function are derived from their representation in many CGE models – the function itself and a first-order optimality condition. In an appendix, these representations are derived from cost minimization and revenue maximization given prices and the “technology” embodied by the Armington and CET functions.

Section 2 covers the Armington function, Section 3 the CET function. The key results are summarized and discussed in Section 4. In two appendices, the first-order conditions related to the Armington and CET functions (as commonly depicted in CGE models) are derived from demand and supply side optimization problems.

2. ARMINGTON FUNCTION AND ELASTICITIES

2.1. Notation

A	shift parameter
d	quantity demanded of domestic output
m	quantity demanded of imports

- q utility (“composite demand”, aggregation of import and domestic demands)
- pd price of domestic output
- pm price of imports
- sm composite demand spending share on imports¹
- δ share parameter
- ρ exponent
- σ elasticity of substitution

2.2. Common formulation in CGE models²

The Armington function:

$$q = A \left(\delta m^{-\rho} + (1 - \delta) d^{-\rho} \right)^{-1/\rho} \quad (2.2.1)$$

The optimal import – domestic-demand ratio:

$$\frac{m}{d} = \left(\frac{pd}{pm} \frac{\delta}{1 - \delta} \right)^{\sigma} = \left(pd \frac{\delta}{1 - \delta} \right)^{\sigma} pm^{-\sigma} \quad (2.2.2)$$

The relationships between the exponent ρ and the elasticity of substitution, σ , maybe written as follows:

$$\sigma = \frac{1}{1 + \rho} \rightarrow 1 + \rho = \frac{1}{\sigma} \rightarrow \rho = \frac{1}{\sigma} - 1 = \frac{1 - \sigma}{\sigma}$$

¹ i.e. $sm = \frac{pm \cdot m}{pm \cdot m + pd \cdot d}$

² See for example Annabi et al. (2006, p. 10), Dervis et al. (1982, p. 222), Lofgren et al. (2002, pp. 25 and 29).

2.3. Alternative 1: Price elasticity of import demand with fixed domestic demands

This definition treats domestic demands, d , as fixed; given this, the adjustment in m gives rise to a change in spending and utility, q . For this case, the price elasticity is simply the negative of the (positive) substitution elasticity. Rearranging (2.2.2):

$$m = d \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma} \quad (2.3.1)$$

Using (2.3.1) to define the price elasticity:

$$\frac{\partial m}{\partial pm} = (-\sigma) \cdot d \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma-1} \quad (2.3.2)$$

$$\frac{m}{pm} = \frac{d \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma}}{pm} = d \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma-1} \quad (2.3.3)$$

and combining the two:

$$\varepsilon_{pm}^m = \frac{\frac{\partial m}{\partial pm}}{\frac{m}{pm}} = \frac{(-\sigma) \cdot d \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma-1}}{d \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma-1}} = -\sigma \quad (2.3.4)$$

2.4. Alternative 2: Price elasticity of import demand with fixed utility -- Hicksian

This elasticity definition is the same as alternative 1 except for the fact that, instead of having domestic demands (d) fixed, utility (q) is fixed. The definition considers the link between imports and domestic purchases (m and d) for a given level of q . In this context, an increase (decrease) in m leads to a decrease (increase) in d . d is defined as the difference between q and m ; in the neighborhood of the calibrated base-year solution, this is correct. The “utility-compensated” price elasticity of import demand may be derived as follows. Using 2.3.1 and the definition of domestic demands as the difference between total demands and imports, i.e.

$$d = \bar{q} - m$$

import demands may be redefined as follows, using expression (2.2.2):

$$\begin{aligned}
m &= (\bar{q} - m) \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma} \\
&= \bar{q} \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma} - m \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma}; \\
m \left(1 + \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma} \right) &= \bar{q} \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma}; \\
m &= \frac{\bar{q} \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma}}{1 + \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma}} \quad (2.4.1)
\end{aligned}$$

Taking the natural log of (2.4.1) and finding its derivative with respect to pm :

$$\begin{aligned}
\ln m &= \ln \bar{q} + \sigma \ln \left(pd \frac{\delta}{(1-\delta)} \right) - \sigma \ln pm - \ln \left(1 + \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma} \right) \\
\frac{\partial \ln m}{\partial pm} &= -\frac{\sigma}{pm} - \frac{(-\sigma) \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma-1}}{1 + \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma}} = -\sigma \left(\frac{1}{pm} - \frac{\left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma-1}}{1 + \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma}} \right) \quad (2.4.2)
\end{aligned}$$

The above derivative fits in the definition of the price elasticity of import demand, ε_{pm}^m , in the following manner:

$$\begin{aligned}
\frac{\partial \ln m}{\partial pm} &= \frac{\frac{\partial m}{\partial pm}}{m} \rightarrow \frac{\partial m}{\partial pm} = \frac{\partial \ln m}{\partial pm} \cdot m \\
\varepsilon_{pm}^m &= \frac{\partial m}{\partial pm} \frac{pm}{m} = \left(\frac{\partial \ln m}{\partial pm} \cdot m \right) \frac{pm}{m} = \left(\frac{\partial \ln m}{\partial pm} \cdot pm \right) \quad (2.4.3)
\end{aligned}$$

Given (2.4.2) and (2.4.3), the price elasticity of import demand may be defined as:

$$\begin{aligned}\varepsilon_{pm}^m &= \left(\frac{\partial \ln m}{\partial pm} \right) pm = -\sigma \left(\frac{1}{pm} - \frac{\left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma-1}}{1 + \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma}} \right) pm \\ &= -\sigma \left(1 - \frac{\left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma}}{1 + \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma}} \right) = -\sigma \left(1 - \frac{\frac{m}{d}}{1 + \frac{m}{d}} \right) = -\sigma \left(\frac{1 + \frac{m}{d} - \frac{m}{d}}{1 + \frac{m}{d}} \right) = -\sigma \left(\frac{1}{1 + \frac{m}{d}} \right) \quad (2.4.4)\end{aligned}$$

Compared to (2.3.4), (2.4.4) includes an adjustment term according to which, for a given elasticity of substitution, the higher the ratio between imports and domestic purchases, the lower the price elasticity of import demand. The smaller this ratio, the closer the absolute value of the price elasticity of demand to the substitution elasticity.

Alternatively, the following definition of the substitution elasticity may be derived from (2.4.4):

$$\sigma = -\varepsilon_{pm}^m \left(1 + \frac{m}{d} \right) \quad (2.4.5)$$

2.5 Alternative 3: Price elasticity of import demand with fixed income -- Marshallian

This definition is the same as alternative 2, (2.3.4), except for the fact that, instead of keeping utility (q) fixed, it keeps spending (y) fixed – for a given y , an increase (decrease) in m leads to a decrease (increase) in d due to the budget constraint. This yields a Marshallian price elasticity of import demand. This is a derivation. Given

$$\bar{y} = pm \cdot m + pd \cdot d$$

domestic demands may be defined as:

$$d = \frac{\bar{y} - pm \cdot m}{pd} \quad (2.5.1)$$

Using (2.5.1) to substitute for d in (2.3.1), the rearranged optimality condition, and solving for m :

$$\begin{aligned}
m &= d \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma} = \left(\frac{\bar{y} - pm \cdot m}{pd} \right) \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma} \\
&= \left(\frac{\bar{y}}{pd} \right) \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma} - m \left(\frac{pm}{pd} \right) \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma}; \\
m \left(1 + \left(\frac{pm}{pd} \right) \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma} \right) &= \left(\frac{\bar{y}}{pd} \right) \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma}; \\
m &= \frac{\left(\frac{\bar{y}}{pd} \right) \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma}}{1 + \left(\frac{pm}{pd} \right) \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma}} \quad (2.5.2)
\end{aligned}$$

Taking the natural log of 2.5.2:

$$\begin{aligned}
\ln m &= \ln \left(\frac{\bar{y}}{pd} \right) + \sigma \ln \left(pd \frac{\delta}{(1-\delta)} \right) - \sigma \ln pm - \ln \left(1 + \left(\frac{pm}{pd} \right) \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma} \right) \\
&= \ln \left(\frac{\bar{y}}{pd} \right) + \sigma \ln \left(pd \frac{\delta}{(1-\delta)} \right) - \sigma \ln pm - \ln \left(1 + \left(\frac{1}{pd} \right) \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma+1} \right) \quad (2.5.3)
\end{aligned}$$

Using 2.5.3, the derivative of the natural log with respect to the import price may be defined as follows:

$$\frac{\partial \ln m}{\partial pm} = -\frac{\sigma}{pm} - \frac{(1-\sigma) \left(\frac{1}{pd} \right) \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma}}{1 + \left(\frac{1}{pd} \right) \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma+1}} \quad (2.5.4)$$

The above derivative fits in the definition of the price elasticity of import demand, ε_{pm}^m , in the following manner:

$$\frac{\partial \ln m}{\partial pm} = \frac{\frac{\partial m}{\partial pm}}{m} \rightarrow \frac{\partial m}{\partial pm} = \frac{\partial \ln m}{\partial pm} \cdot m$$

$$\varepsilon_{pm}^m = \frac{\partial m}{\partial pm} \frac{pm}{m} = \left(\frac{\partial \ln m}{\partial pm} \cdot m \right) \frac{pm}{m} = \left(\frac{\partial \ln m}{\partial pm} \cdot pm \right) \quad (2.5.5)$$

Substituting (2.5.4) into (2.5.5):

$$\varepsilon_{pm}^m = \left(-\frac{\sigma}{pm} - \frac{(1-\sigma) \left(\frac{1}{pd} \right) \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma}}{1 + \left(\frac{1}{pd} \right) \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma+1}} \right) pm = \left(-\sigma - \frac{(1-\sigma) \left(\frac{1}{pd} \right) \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma+1}}{1 + \left(\frac{1}{pd} \right) \left(pd \frac{\delta}{(1-\delta)} \right)^\sigma pm^{-\sigma+1}} \right)$$

$$= \frac{-\sigma - \sigma \left(\frac{pm}{pd} \right) \left(\frac{pd}{pm} \frac{\delta}{(1-\delta)} \right)^\sigma - \left(\frac{pm}{pd} \right) \left(\frac{pd}{pm} \frac{\delta}{(1-\delta)} \right)^\sigma + \sigma \left(\frac{pm}{pd} \right) \left(\frac{pd}{pm} \frac{\delta}{(1-\delta)} \right)^\sigma}{1 + \left(\frac{pm}{pd} \right) \left(\frac{pd}{pm} \frac{\delta}{(1-\delta)} \right)^\sigma}$$

$$= \frac{-\sigma - \left(\frac{pm}{pd} \right) \left(\frac{pd}{pm} \frac{\delta}{(1-\delta)} \right)^\sigma}{1 + \left(\frac{pm}{pd} \right) \left(\frac{pd}{pm} \frac{\delta}{(1-\delta)} \right)^\sigma} = \frac{-\sigma - \left(\frac{pm}{pd} \right) \left(\frac{m}{d} \right)}{1 + \left(\frac{pm}{pd} \right) \left(\frac{m}{d} \right)} = \frac{-\sigma - \left(\frac{pm \cdot m}{pd \cdot d} \right)}{1 + \left(\frac{pm \cdot m}{pd \cdot d} \right)} \quad (2.5.6)$$

According to Ramskov and Munksgaard (2001, p. 11) and Shoven and Whalley (1992, p. 96) the same own-price elasticity of demand, here referred to as ε_{pm}^{m2} , may be written as follows, adapted to match our context and notation:

$$\varepsilon_{pm}^m = -\sigma + (\sigma - 1) \left(\frac{pm \cdot m}{pm \cdot m + pd \cdot d} \right) = -\sigma + (\sigma - 1) s_m = \varepsilon_{pm}^{m2} \quad (2.5.7)$$

The two definitions, (2.5.6) and (2.5.7) are consistent. Deriving ε_{pm}^{m2} (2.5.7) from ε_{pm}^m (2.5.6):

$$\begin{aligned}
\varepsilon_{pm}^m &= \frac{-\sigma - \left(\frac{pm \cdot m}{pd \cdot d}\right)}{1 + \left(\frac{pm \cdot m}{pd \cdot d}\right)} = -\frac{\sigma}{1 + \left(\frac{pm \cdot m}{pd \cdot d}\right)} - \frac{\left(\frac{pm \cdot m}{pd \cdot d}\right)}{1 + \left(\frac{pm \cdot m}{pd \cdot d}\right)} = -\frac{\sigma}{\left(\frac{pd \cdot d}{pd \cdot d}\right) + \left(\frac{pm \cdot m}{pd \cdot d}\right)} - \frac{\left(\frac{pm \cdot m}{pd \cdot d}\right)}{\left(\frac{pd \cdot d}{pd \cdot d}\right) + \left(\frac{pm \cdot m}{pd \cdot d}\right)} \\
&= -\frac{\sigma}{\left(\frac{pd \cdot d + pm \cdot m}{pd \cdot d}\right)} - \frac{\left(\frac{pm \cdot m}{pd \cdot d}\right)}{\left(\frac{pd \cdot d + pm \cdot m}{pd \cdot d}\right)} = -\sigma \left(\frac{pd \cdot d}{pd \cdot d + pm \cdot m}\right) - \frac{pm \cdot m}{pd \cdot d + pm \cdot m} \\
&= -\sigma \left(\frac{pd \cdot d + pm \cdot m - pm \cdot m}{pd \cdot d + pm \cdot m}\right) - \frac{pm \cdot m}{pd \cdot d + pm \cdot m} = -\sigma \left(\frac{pd \cdot d + pm \cdot m}{pd \cdot d + pm \cdot m} - \frac{pm \cdot m}{pd \cdot d + pm \cdot m}\right) - \frac{pm \cdot m}{pd \cdot d + pm \cdot m} \\
&= -\sigma \left(1 - \frac{pm \cdot m}{pd \cdot d + pm \cdot m}\right) - \frac{pm \cdot m}{pd \cdot d + pm \cdot m} = -\sigma + \sigma \left(\frac{pm \cdot m}{pd \cdot d + pm \cdot m}\right) - \frac{pm \cdot m}{pd \cdot d + pm \cdot m} \\
&= -\sigma + (\sigma - 1) \left(\frac{pm \cdot m}{pd \cdot d + pm \cdot m}\right) = -\sigma + (\sigma - 1) s_m = \varepsilon_{pm}^{m2}
\end{aligned}$$

(2.5.7) may be rearranged to yield the following formula for computing the elasticity of substitution, σ : given an available estimate for the price elasticity of import demand:

$$\varepsilon_{pm}^m = -\sigma + \sigma \cdot sm - sm;$$

$$\varepsilon_{pm}^m + sm = \sigma \cdot sm - \sigma = \sigma(sm - 1);$$

$$\sigma = \frac{\varepsilon_{pm}^m + sm}{sm - 1} \quad (2.5.8)$$

3. CET FUNCTION AND ELASTICITIES

3.1. Notation

- A shift parameter
- d quantity of supply to domestic markets
- e quantity of exports
- q output
- pd price of domestic output
- pe price of exports

- δ share parameter
- ρ exponent
- Ω elasticity of transformation

Note that the interpretations of d , e , and q are different in the CET context.

3.2. Common formulation in CGE Models³

CET function:

$$q = A(\delta e^\rho + (1 - \delta)d^\rho)^{1/\rho} \quad (3.2.1)$$

The optimal export-domestic supply ratio:

$$\frac{e}{d} = \left(\frac{pe(1-\delta)}{pd\delta} \right)^\Omega \quad (3.2.2)$$

The relationship between exponent and transformation elasticity:

$$\rho = \frac{1}{\Omega} + 1 \rightarrow \rho - 1 = \frac{1}{\Omega} \rightarrow \Omega = \frac{1}{\rho - 1}$$

3.3. Price elasticity of export supply with fixed domestic supply

This definition treats the domestic supply, d , as fixed; given this, the adjustment in e gives rise to a change in output, q . For this case, the price elasticity is simply the transformation elasticity. Rearranging (3.2.2) and taking the derivative with respect to pe :

$$e = d \left(\frac{1}{pd} \frac{(1-\delta)}{\delta} \right)^\Omega pe^\Omega \quad (3.3.1)$$

$$\frac{e}{pe} = \frac{d \left(\frac{1}{pd} \frac{(1-\delta)}{\delta} \right)^\Omega pe^\Omega}{pe} = d \left(\frac{1}{pd} \frac{(1-\delta)}{\delta} \right)^\Omega pe^{\Omega-1} \quad (3.3.2)$$

³ See for example Annabi et al. (2006, p. 10) and Lofgren et al. (2002, p. 28).

Taking the derivative of (3.3.1) with respect to pe :

$$\frac{\partial e}{\partial pe} = \Omega \cdot d \cdot \left(\frac{1}{pd} \frac{(1-\delta)}{\delta} \right)^{\frac{1}{\rho-1}} pe^{\Omega-1} \quad (3.3.3)$$

Recalling the elasticity formula and plugging (3.3.2) and (3.3.3) into this formula:

$$\varepsilon_{pe}^e = \frac{\frac{\partial e}{\partial pe}}{\frac{e}{pe}} = \frac{\Omega \cdot d \cdot \left(\frac{1}{pd} \frac{(1-\delta)}{\delta} \right)^{\frac{1}{\rho-1}} pe^{\Omega-1}}{d \left(\frac{1}{pd} \frac{(1-\delta)}{\delta} \right)^{\frac{1}{\rho-1}} pe^{\Omega-1}} = \Omega \quad (3.3.4)$$

The above definition is the same as in de Melo and Tarr (1992, pp. 231-232).

3.4. Price elasticity of export supply with fixed q

This definition is the same as alternative 1 except for the fact that it considers the link between output, exports, and domestic supplies (q , e , and d) – for a given q , an increase (decrease) in e leads to a decrease (increase) in d . d is defined as the difference between q and e ; in the neighborhood of the calibrated base-year solution, this is correct; alternatively (perhaps better), one could define d using the CET function (what results when it is solved for d) – here $\bar{q} - e$ is used.⁴ Given this definition of domestic supplies, d , a rearranged version of (3.2.2) may be used to define e as follows:

$$^4 q = A(\delta e^\rho + (1-\delta)d^\rho)^{1/\rho} \rightarrow q^\rho = A\delta e^\rho + A(1-\delta)d^\rho \rightarrow$$

$$d^\rho = \frac{q^\rho - A\delta e^\rho}{A(1-\delta)} \rightarrow d = \left(\frac{q^\rho - A\delta e^\rho}{A(1-\delta)} \right)^{1/\rho}$$

$$\begin{aligned}
e &= d \left(\frac{1}{pd} \frac{(1-\delta)}{\delta} \right)^\Omega pe^\Omega = (\bar{q} - e) \left(\frac{1}{pd} \frac{(1-\delta)}{\delta} \right)^\Omega pe^\Omega \\
&= \bar{q} \left(\frac{1}{pd} \frac{(1-\delta)}{\delta} \right)^\Omega pe^\Omega - e \left(\frac{1}{pd} \frac{(1-\delta)}{\delta} \right)^\Omega pe^\Omega; \\
e \left(1 + \left(\frac{1}{pd} \frac{(1-\delta)}{\delta} \right)^\Omega pe^\Omega \right) &= \bar{q} \left(\frac{1}{pd} \frac{(1-\delta)}{\delta} \right)^\Omega pe^\Omega; \\
e &= \frac{\bar{q} \left(\frac{1}{pd} \frac{(1-\delta)}{\delta} \right)^\Omega pe^\Omega}{1 + \left(\frac{1}{pd} \frac{(1-\delta)}{\delta} \right)^\Omega pe^\Omega} \quad (3.4.1)
\end{aligned}$$

Defining the natural log of (3.4.1) and taking its derivative with respect to pe :

$$\begin{aligned}
\ln e &= \ln \bar{q} + \Omega \ln \left(\frac{1}{pd} \frac{(1-\delta)}{\delta} \right)^\Omega + \Omega \ln pe - \ln \left(1 + \left(\frac{1}{pd} \frac{(1-\delta)}{\delta} \right)^\Omega pe^\Omega \right) \\
\frac{\partial \ln e}{\partial pe} &= \frac{\Omega}{pe} - \frac{\Omega \left(\frac{1}{pd} \frac{(1-\delta)}{\delta} \right)^\Omega pe^{\Omega-1}}{1 + \left(\frac{1}{pd} \frac{(1-\delta)}{\delta} \right)^\Omega pe^\Omega} = \Omega \left(\frac{1}{pe} - \frac{\left(\frac{1}{pd} \frac{(1-\delta)}{\delta} \right)^\Omega pe^{\Omega-1}}{1 + \left(\frac{1}{pd} \frac{(1-\delta)}{\delta} \right)^\Omega pe^\Omega} \right) \quad (3.4.2)
\end{aligned}$$

The above derivative fits into this definition of the price elasticity of export supply, ε_{pe}^e , in the following manner:

$$\begin{aligned}
\frac{\partial \ln e}{\partial pe} &= \frac{\frac{\partial e}{e}}{\frac{\partial pe}{e}} \\
\frac{\partial e}{\partial pe} &= \frac{\partial \ln e}{\partial pe} \cdot e; \\
\varepsilon_{pe}^e &= \frac{\partial e}{\partial pe} \frac{pe}{e} = \left(\frac{\partial \ln e}{\partial pe} \cdot e \right) \frac{pe}{e} = \left(\frac{\partial \ln e}{\partial pe} \cdot pe \right) \\
&= \Omega \left(\frac{1}{pe} - \frac{\left(\frac{1}{pd} \frac{(1-\delta)}{\delta} \right)^\Omega pe^{\Omega-1}}{1 + \left(\frac{1}{pd} \frac{(1-\delta)}{\delta} \right)^\Omega pe^\Omega} \right) pe = \Omega \left(1 - \frac{\left(\frac{1}{pd} \frac{(1-\delta)}{\delta} \right)^\Omega pe^\Omega}{1 + \left(\frac{1}{pd} \frac{(1-\delta)}{\delta} \right)^\Omega pe^\Omega} \right) \\
&= \Omega \left(1 - \frac{\left(\frac{qe}{qd} \right)}{1 + \left(\frac{qe}{qd} \right)} \right) = \Omega \left(\frac{1 + \left(\frac{qe}{qd} \right) - \left(\frac{qe}{qd} \right)}{1 + \left(\frac{qe}{qd} \right)} \right) = \frac{\Omega}{1 + \left(\frac{qe}{qd} \right)} \quad (3.4.3)
\end{aligned}$$

The difference between alternatives 1, (3.3.4) and 2 (3.4.3) lies in denominator according to which, the lower the ratio between exports and domestic sales, the higher the price elasticity of export supply for a given elasticity of transformation. As the ratio approaches zero, (3.4.3.) approaches (3.3.4) and the transformation elasticity.

Accordingly, given (3.4.3) and an estimate of the price elasticity of export demand that matches these assumptions, the transformation elasticity may be defined as follows:

$$\Omega = \varepsilon_{pe}^e \left(1 + \left(\frac{qe}{qd} \right) \right) \quad (3.4.4)$$

4. SUMMARY AND CONCLUDING REMARKS

The results are summarized in Tables 4.1 and 4.2. The selection of which formula to use to derive the substitution or transformation elasticities from elasticities of import demand and export supply depends on the assumptions that were applied in the estimation of the latter. This latter observation points to the fact that the selection of elasticity values for a CGE application is not a mechanical procedure. It is important to keep in mind, not only the assumptions made in the econometrics but also the structure and validity of the model. To

exemplify, if a review of simulation results indicate that factors are more mobile in the model than in empirical reality (under the relevant time frame), then sectoral reallocations of production factors and output may lead to excessive responses in exports and imports to changes in relative prices of traded and non-traded commodities – unless the modeler opts to adjust the model treatment of factor markets (which may require data that is not available), then a second-best solution may be to reduce substitution and/or transformation elasticities.

Table 4.1. Formulas for linking Armington elasticities of substitution to import elasticities of demand

Fixed variable	$\varepsilon_{pm}^m = f(\sigma)$	$\sigma = f(\varepsilon_{pm}^m)$
Domestic demand (d)	(2.3.4) $\varepsilon_{pm}^m = -\sigma$	$\sigma = -\varepsilon_{pm}^m$
Utility (q)	(2.4.4) $\varepsilon_{pm}^m = -\sigma / \left(1 + \frac{m}{d}\right)$	(2.4.5) $\sigma = -\varepsilon_{pm}^m \left(1 + \frac{m}{d}\right)$
Spending (y)	(2.5.7) $\varepsilon_{pm}^m = -\sigma + (\sigma - 1)sm$	(2.5.8) $\sigma = \frac{\varepsilon_{pm}^m + sm}{sm - 1}$

Note: For notation, see section 2.1.

Table 4.2. Formulas linking CET elasticities of transformation to export elasticities of supply

Fixed variable	$\varepsilon_{pe}^e = f(\Omega)$	$\Omega = f(\varepsilon_{pe}^e)$
Domestic supply	(3.3.4) $\varepsilon_{pe}^e = \Omega$	$\Omega = \varepsilon_{pe}^e$

(d)		
Output (q)	(3.4.3) $\varepsilon_{pe}^e = \Omega / \left(1 + \frac{qe}{qd}\right)$	(3.4.4) $\Omega = \varepsilon_{pe}^e \left(1 + \frac{qe}{qd}\right)$

Note: For notation, see section 3.1.

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Appendix A: Derivation of the optimal import – domestic purchase combination

The optimality condition for the Armington function may be derived from the problem of minimizing the cost (C) of achieving a given level of utility:

$$\text{Minimize } C = pm \cdot m + pd \cdot d$$

$$\text{subject to } q = A(\delta m^{-\rho} + (1-\delta)d^{-\rho})^{-1/\rho}$$

$$\text{Lagrangian: } L = pm \cdot m + pd \cdot d + \lambda \left(q - A(\delta m^{-\rho} + (1-\delta)d^{-\rho})^{-1/\rho} \right)$$

First-order conditions:

$$(1) \frac{\partial L}{\partial m} = pm - \lambda \left(-\frac{1}{\rho} \right) A(\delta m^{-\rho} + (1-\delta)d^{-\rho})^{-\frac{1}{\rho}-1} (-\rho) \delta m^{-\rho-1} = 0$$

$$(2) \frac{\partial L}{\partial d} = pd - \lambda \left(-\frac{1}{\rho} \right) A(\delta m^{-\rho} + (1-\delta)d^{-\rho})^{-\frac{1}{\rho}-1} (-\rho)(1-\delta)d^{-\rho-1} - \lambda \cdot pd = 0$$

$$(3) \frac{\partial L}{\partial \lambda} = \left(q - A(\delta m^{-\rho} + (1-\delta)d^{-\rho})^{-1/\rho} \right) = 0$$

Rearranging (1) and (2) and taking the ratio:

$$\frac{pd}{pm} = \frac{\left(-\frac{1}{\rho} \right) A(\delta m^{-\rho} + (1-\delta)d^{-\rho})^{-\frac{1}{\rho}-1} (-\rho)(1-\delta)d^{-\rho-1}}{\left(-\frac{1}{\rho} \right) A(\delta m^{-\rho} + (1-\delta)d^{-\rho})^{-\frac{1}{\rho}-1} (-\rho) \delta m^{-\rho-1}}$$

$$\rightarrow \frac{pd}{pm} = \frac{(1-\delta)d^{-\rho-1}}{\delta m^{-\rho-1}} \rightarrow \frac{pd}{pm} \frac{\delta}{(1-\delta)} = \left(\frac{m}{d} \right)^{1+\rho} \rightarrow \frac{m}{d} = \left(\frac{pd}{pm} \frac{\delta}{(1-\delta)} \right)^{1/(1+\rho)}$$

$$\rightarrow (4) \frac{m}{d} = \left(\frac{pd}{pm} \frac{\delta}{(1-\delta)} \right)^{\sigma}$$

The resulting first-order conditions consist of equations 3 and 4.

Appendix B: Derivation of the optimal export - domestic supply combination

The optimality condition for the CET function may be derived from the problem of maximizing the revenue (R) from any given output level:

$$\text{Maximize } R = pe \cdot e + pd \cdot d$$

$$\text{subject to } q = A(\delta e^\rho + (1-\delta)d^\rho)^{1/\rho}$$

$$\text{Lagrangian: } L = pe \cdot e + pd \cdot d + \lambda \left(q - A(\delta e^\rho + (1-\delta)d^\rho)^{1/\rho} \right)$$

$$(1) \frac{\partial L}{\partial e} = pe - \lambda \left(\frac{1}{\rho} \right) A(\delta e^\rho + (1-\delta)d^\rho)^{\frac{1}{\rho}-1} \rho \delta e^{\rho-1} = 0$$

$$(2) \frac{\partial L}{\partial d} = pd - \lambda \left(\frac{1}{\rho} \right) A(\delta e^\rho + (1-\delta)d^\rho)^{\frac{1}{\rho}-1} \rho (1-\delta) d^{\rho-1} = 0$$

$$(3) \frac{\partial L}{\partial \lambda} = q - A(\delta e^\rho + (1-\delta)d^\rho)^{1/\rho} = 0$$

Rearranging (1) and (2) and taking the ratio:

$$\frac{pe}{pd} = \frac{\lambda \left(\frac{1}{\rho} \right) A(\delta e^\rho + (1-\delta)d^\rho)^{\frac{1}{\rho}-1} \rho \delta e^{\rho-1}}{\lambda \left(\frac{1}{\rho} \right) A(\delta e^\rho + (1-\delta)d^\rho)^{\frac{1}{\rho}-1} \rho (1-\delta) d^{\rho-1}} = \frac{\delta e^{\rho-1}}{(1-\delta) d^{\rho-1}} = \frac{\delta}{(1-\delta)} \left(\frac{e}{d} \right)^{\rho-1}$$

$$\rightarrow \frac{pe}{pd} \frac{(1-\delta)}{\delta} = \left(\frac{e}{d} \right)^{\rho-1} \rightarrow \left(\frac{pe}{pd} \frac{(1-\delta)}{\delta} \right)^{\frac{1}{\rho-1}} = \left(\frac{e}{d} \right)^{\frac{\rho-1}{\rho-1}} \rightarrow \frac{e}{d} = \left(\frac{pe}{pd} \frac{(1-\delta)}{\delta} \right)^{\frac{1}{\rho-1}}$$

$$\rightarrow (4) \frac{e}{d} = \left(\frac{pe}{pd} \frac{(1-\delta)}{\delta} \right)^{\frac{1}{\rho-1}}$$

The resulting first-order conditions consist of equations 3 and 4.